Algebraic Geometry: Lecture 3

Zariski Topology.

Given a set X, a *topology* on X is just a list T of subsets of X that satisfy the following properties:

(1) $\varnothing \in T$, $X \in T$

(2) If $A_1, A_2, \ldots \in T$ then $\bigcup_i A_i \in T$

(3) If $A, B \in T$ then $A \cap B \in T$.

The subsets of X that belong to T are called the *open sets of* X.

If $A \subset X$ is an open set then $X \setminus A$ is called a *closed set*. So we can just as easily define a topology on a set X by listing all the closed sets, and then taking the open sets to be all their complements.

As last week, assume k is an algebraically closed field. Recall that for a subset $S \subset k[X_1, \ldots, X_n]$,

$$V(S) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in S\},$$

and these sets are called the affine algebraic sets.

The Zariski topology is just a topology on $\mathbb{A}^n$ where the closed sets are precisely the algebraic sets in $\mathbb{A}^n$. It's an easy exercise to see this is a topology using the facts that:

(1) $\varnothing = V(k[X_1, \ldots, X_n])$, $\mathbb{A}^n = V(0)$,

(2) $V(S_1) \cup V(S_2) = V(S_1 S_2)$,

(3) $\bigcap_i V(S_i) = V\left(\sum_i S_i\right)$.

The Zariski topology isn't very subtle. Closed sets are mostly very small, for example if $k = \mathbb{C}$ then a typical closed set is just a finite set of points, hence a typical open set is all of $\mathbb{C}$ except a finite number of points.

The Zariski topology on $\mathbb{P}^n$ is defined exactly the same way, with projective algebraic sets forming the closed sets. Other than the fact you're now dealing with homogeneous polynomials, everything is the same.

Using this topology we define a quasi-affine variety to be an open subset of an affine variety, and a quasi-projective variety to be an open subset of a projective variety.

Functions on Varieties

Polynomial functions.

Let $V \subset \mathbb{A}^n$ be an algebraic set and $I(V)$ the corresponding ideal. (Recall, $I(V)$ is the ideal of polynomials that vanish at all points of V.)

We define a *polynomial function* on V to be a map $f : V \to k$ of the form $P \mapsto F(P)$ where $F \in k[X_1, \ldots, X_n]$. So f is the restriction of a polynomial $F : \mathbb{A}^n \to k$. By definition of $I(V)$, two polynomials $F, G \in k[X_1, \ldots, X_n]$ define the same function on V if and only if $F(P) - G(P) = 0$ for all $P \in V$, i.e. if and only if $F - G \in I(V)$. So we define the *coordinate ring* $k[V]$ by

$$k[V] = \{f : V \to k \mid f \text{ is a polynomial function } \} \cong k[X_1, \ldots, X_n]/I(V).$$

Polynomials maps.

A generalisation of the above idea is as follows.

Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic sets. A map $f : V \to W$ is a *polynomial map* if there exist m polynomials $F_1, \ldots, F_m \in k[X_1, \ldots, X_n]$ such that

$$f(P) = (F_1(P), \ldots, F_m(P)) \in W \qquad \text{for all } P \in V.$$

In particular a polynomial function is just a polynomial map with $m = 1$.

Examples

(1) Simple parameterisations like $\mathbb{C} \to \mathbb{C}^2, t \mapsto (t^2, t^3)$ (a cuspidal cubic), or $t \mapsto (t^2 - 1, t^3 - 1)$ (nodal cubic) are polynomial maps.

(2) We can also take projections, for example $\pi : \mathbb{C}^3 \to \mathbb{C}^2, (x, y, z) \mapsto (x, y)$.

A polynomial map $f : V \to W$ between algebraic sets is called an isomorphism if there exists a polynomial map $g : W \to V$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$.

An *affine variety* is an irreducible algebraic subset $V \subset \mathbb{A}^n$, defined up to isomorphism. If V is an affine variety then we saw last week that $I(V)$ is a prime ideal, which means $k[V] = k[X_1, \ldots, X_n]/I(V)$ is an integral domain. So we can define...

The *function field* $k(V)$ of V is the field of fractions

$$k(V) = \mathrm{Frac}\,(k[V]) = \left\{ \frac{g}{h} \,\middle|\, g, h \in k[V], h \neq 0 \right\}.$$

An element $f \in k(V)$ is called a *rational function* on V. f is not really a function on V because its denominator will probably have zeroes. But away from these places it is a function, which motivates...

Let $f \in k(V)$ and $P \in V$. We say f is *regular* at P if there exist an expression $f = g/h$ with $g, h \in k[V]$ and $h(P) \neq 0$. If f is regular at all points of V then we simply say it is regular.

Example

Let $V = \{XT - YZ = 0\} \subset \mathbb{A}^4$. Consider the rational function $f = X/Y \in k(V)$. Then f is regular at the point $(X, Y, Z, T) = (0, 0, 1, 1)$. This is because even though we get $X/Y = 0/0$, we can also write $f = Z/T$, and at this point $T \neq 0$.

We write

$$\operatorname{dom} f = \{P \in V \mid f \text{ is regular at } P\}$$

for the *domain of definition* of f.

We also define

$$\begin{aligned} \mathcal{O}_{V,P} &= \{f \in k(V) \mid f \text{ is regular at } P\} \\ &= k[V]\left[h^{-1} \mid h(P) \neq 0\right]. \end{aligned}$$

$\mathcal{O}_{V,P} \subset k(V)$ is a subring called the *local ring* of V at P.

Rational maps.

Given an affine variety V, a *rational map* $f : V \dashrightarrow \mathbb{A}^n$ is a partially defined map given by rational functions $f_1, \ldots, f_n \in k(V)$, i.e.

$$f(P) = (f_1(P), \ldots, f_n(P)) \qquad \text{for all } P \in \bigcap_{i=1}^{n} \operatorname{dom} f_i.$$

By definition $\operatorname{dom} f = \bigcap \operatorname{dom} f_i$ and f is called regular at $P \in V$ if and only if $P \in \operatorname{dom} f$.

A rational map $V \dashrightarrow W$ between two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is defined to be a rational map $f : V \dashrightarrow \mathbb{A}^m$ such that $f(\operatorname{dom} f) \subset W$, i.e. its image, where defined, is contained in W.

Example

There is a rational map from $\mathbb{R}$ to the circle given by

$$f : \lambda \mapsto \left(\frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2-1}{\lambda^2+1}\right).$$

f is regular for all $\lambda \in \mathbb{R}$.

Composition of rational maps may not always be defined. The problem is that the composite $g \circ f$ is defined on $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$, and this can easily be empty.

Example

Let $f : \mathbb{A}^1 \to \mathbb{A}^2 : X \mapsto (X, 0)$ and $g : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1 : (X, Y) \mapsto \frac{X}{Y}$. So $\operatorname{dom} f = \mathbb{A}^1$, $\operatorname{dom} g = \{\mathbb{A}^2 \mid Y \neq 0\}$, and $f^{-1}(\operatorname{dom} g) = \varnothing$.

We can sort out which maps allow compositions with the following definition.

A rational map $f : V \dashrightarrow W$ is *dominant* if $f(\operatorname{dom} f)$ is dense in W for the Zariski topology. That is, if we pick any point $Q \in W$ and any open set S containing Q, then S will also contain an element of $f(\operatorname{dom} f)$.

Using the fact that rational maps are continuous we get that $f^{-1}(\operatorname{dom} g) \subset \operatorname{dom} f$ is a dense open set for any rational map $g : W \dashrightarrow U$, so $g \circ f$ is defined on a dense open set of V, so $g \circ f : V \dashrightarrow U$ is at least partially defined.

Morphisms.

Given an open set $U \subset V$, a *morphism* $f : U \to W$ is a rational map $f : V \dashrightarrow W$ such that $U \subset \operatorname{dom} f$, so that f is regular at every $P \in U$.

If $U_1 \subset V$ and $U_2 \subset W$ are both open then a morphism $f : U_1 \to U_2$ is just a morphism $f : U_1 \to W$ such that $f(U_1) \subset U_2$.

An isomorphism is a morphism that has an inverse morphism, i.e. a morphism f for which there exists a morphism g with $f \circ g$ and $g \circ f$ both being identity maps.

Projective things.

Most things in the projective case are entirely analogous to the affine case.

If $V \subset \mathbb{P}^n$ is an irreducible projective algebraic set then a *rational function* on V is a partially defined function $f : V \dashrightarrow k$ given by $f(P) = g(P)/h(P)$ where $g, h \in k[X_0, \ldots, X_n]$ are homogeneous polynomials of the same degree.

Clearly g/h and g'/h' define the same rational function on V if and only if $h'g - g'h \in I(V)$, so

$$k(V) = \left\{ \frac{g}{h} \,\middle|\, g, h \in k[X_0, \ldots, X_n] \text{ are homogeneous polynomials of the same degree}, h \notin I(V) \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation

$$\frac{g}{h} \sim \frac{g'}{h'} \quad \iff \quad h'g - g'h \in I(V).$$

$k(V)$ is called the *function field* of V.

The definitions of a rational function being regular at P, $\operatorname{dom} f$, and $\mathcal{O}_{V,P}$ are identical to the affine case.

If $V \subset \mathbb{P}^n$ then a *rational map* $V \dashrightarrow \mathbb{P}^m$ is defined by

$$P \mapsto [f_0(P), f_1(P), \ldots, f_m(P)]$$

where $f_0, \ldots, f_m \in k(V)$. This gives the same map as

$$P \mapsto [g(P)f_0(P), g(P)f_1(P), \ldots, g(P)f_m(P)]$$

for any nonzero $g \in k(V)$. In particular if f_0 is never zero then we may assume that $f_0 \equiv 1$.

A rational map $f : V \dashrightarrow \mathbb{P}^m$ is *regular* at $P \in V$ if there exists an expression $f = (f_0, \ldots, f_m)$ such that

(1) $f_0, \ldots, f_m$ are all regular at P,

(2) at least one $f_i(P) \neq 0$.

Again, if $U \subset V$ is an open subset of a projective variety V then a morphism $f : U \to W$ is a rational map $f : V \to W$ with $U \subset \operatorname{dom} f$.

Birational maps.

Let V and W be (affine or projective) varieties. A rational map $f : V \dashrightarrow W$ is called *birational* if it has a rational inverse, i.e. if there is a rational map $g : W \dashrightarrow V$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$.